Fuzzy stochastic optimization: an overview

Stanislas Sakera RUZIBIZA

Independent Institute of Lay Adventists of Kigali (INILAK), P.O. BOX 6392

Kigali, Rwanda

Email : ruzibizass2000@yahoo.com

Abstract

Fuzzy stochastic Optimization deals with situations where fuzziness and randomness co-occur in an optimization setting.

In this paper, we take a general look at core ideas that make up the burgeoning body of Fuzziness Stochastic Optimization, emphasizing the methodological view.

Being a survey, the paper includes many references to both give due credit to results obtained in this field and to help readers get more detailed information on issues of interest.

Keywords: Fuzzy, stochastic, optimization

Introduction

Optimization is a very old and classical area which is of high concern to many disciplines.

Engineering as well as management, Politics as well as Medicine, Artificial Intelligence as well as Operations Research and many other fields are in one way or another concerned with optimization of design, decisions, structures, procedure or information processes.

Most of Optimization problems encountered in Operations Research are essentially based on the homo-economicus model. They consist of maximizing or minimizing a utility function, reflecting decision maker's preferences, under some constraints, expressing decision maker's restrictions.

Analysis of such problems along with construction of algorithms for solving them constitutes the disciplinary matrix of Mathematical programming.

Optimization's theoretical underpinning is now well established and as a result, a broader array of techniques including the simplex method (S.I. Gass, 1985), ellipsoid method (L.G.Khachiayan, 1979), gradient projection methods (M.Avriel, 1976), cutting-plane methods (B.C.Eaves and W.I. Zangwill, 1971) have been developed.

User friendly software with powerful computational and visualization capabilities have also been pushed forward.

All these methods rely heavily on the assumption that involved parameters have well-known fixed values and then take advantage of inherent computational convenience.

Unfortunately, most concrete real-life problems involve some level of uncertainty about values to be assigned to various parameters or about layout of some of the problem's components.

When a probabilistic description of unknown elements is at hand, one is naturally lead to Stochastic Optimization (S.Vajda, 1972, J.K. Sengupta, 1972, J.Gentle, W.Härdle and Y. Mari, 2004).

In the presence of intrinsic or informational imprecision, one has to resort to Fuzzy Optimization (M.K. Luhandjula, 1989, D. Dubey , S. Chandra, 2012 and D. Dubois, 2011).

Nevertheless, in some significant real life problems, one has to base decisions on information which is both fuzzily imprecise and probabilistically uncertain (S. Wang and J. Watada, 2012, S. Wang, G.H.Huang and B.T. Yang, 2012). Fuzzy Stochastic Optimization provides a glimpse into jostling with this kind of problems.

The purpose of this paper is to convey essential information on the field of Fuzzy Stochastic Optimization to broad audience in a way to foster a cross-fertilization of ideas in this field.

The general aim of Fuzzy Stochastic Optimization is to deal with situations where fuzziness and randomness are under one roof in an optimization framework.

The term can encompass many diverse models and therefore means different things to different people. In this paper we review some aspects of Fuzzy Stochastic Optimization potentially of interest to a broad audience.

We shall restrict ourselves to linear optimization problems so that the main ideas are illustrated in a simpler context.

The remaining of this paper is organized as follows: in Section 2, we discuss flexible programming problems with random data. Section 3 is devoted to Mathematical programming problems with random variables having fuzzy parameters. In Section 4, we address mathematical programming problems with fuzzy random coefficients. Extensions and applications of ideas discussed are presented in Section 5. We end up in Section 6 with concluding remarks along with perspectives for future research.

Flexible programming with random data

In this section we focus on situations where the objective function of a stochastic program, as well as its constraints, is not strictly imperatives.

Some leeways may be accepted in their fulfillment.

This leads to a problem of the type:

$$(P_1)\begin{cases}\widetilde{\min}\ \tilde{c}x\\ \widetilde{A}_i x \lesssim \bar{b}_i; \quad i = 1, \dots, m\\ x \in X = \{x \in \mathbb{R}^n / x \geq 0\}\end{cases}$$

where " $\sim$ " means flexibility and "$-$" means the datum is random.

This model has been addressed in literature (M.K. Luhandjula, 1983 and E. Czogala, 1988) by taking advantage of both Fuzzy Set Theory (D. Dubois and H. Pragde, 1980) and Stochastic Optimization.

Here is quintessential of ideas developed to cope (P_1).

First and foremost it has been noted that (P_1) is an ill-defined problem. Both the notion of optimum and the pure rationality principle (P. Kall and S.W. Wallace, 1994) no longer apply.

Researchers interested with this problem resorted them to Simon's bounded rationality principle and sought for satisfying solution rather than an optimal one.

After having put the objective function of (P_1) in the following constraint form:

$$\overline{C}\, x \leq C_0 ,$$

where C_0 is a threshold fixed by the decision-maker, (P_1) reads merely:

Find $x \in X$ such that:

$$\{ \widetilde{A}_i x \lesssim \bar{b}_i; \quad i = 0,1, \dots, m \qquad (1)$$

where $A_0 = \overline{C}$ and $\bar{b}_0 = C_0$

Each inequality of system (1) is then represented as a probabilistic set (K. Hirota, 1981) on $(\Omega, \mathcal{F}, P)$ with membership function with membership function $\mu_i(x, \omega)$ that may be defined as follows:

$$\mu_i(x,\omega) = \begin{cases}1 & \text{if } \bar{A}_i(x,\omega) \leq b_i(\omega)\\ 1 - \frac{\bar{A}_i(\omega)x - b_i(\omega)}{d_i} & \text{if } \bar{b}_i(\omega) < \bar{A}_i(\omega)x \leq b_i(\omega) + d_i\\ 0 & \text{if } \bar{A}_i(\omega)x > b_i(\omega) + d_i\end{cases}$$

where $d_i > 0$ is a constant chosen by the decision maker for a permitted violation of constraint i.

It is worth mentioning that $\mu_i(x,\omega)$ is the level to which the constraint:

$$\bar{A}_i(x,\omega) \leq b_i(\omega) \quad \omega \in \Omega$$

is satisfied.

Other kinds of membership function, more appropriate to the situation at hand may be used instead of the above piecewise linear functions

$$\bar{b}_i(\omega) < \bar{A}_i(\omega)x \leq b_i(\omega) + d_i$$

$$\bar{A}_i(\omega)x > b_i(\omega) + d_i$$

The most used are the logistic and hyperbolic functions (P. Vasant, R. Nagarajan and S. Yaacob, 2005).

According to Bellman-Zadeh's confluence principle (A. Charnes and W.W. Cooper, 1963), a decision in a fuzzy environment is an option that is at the intersection of fuzzy goals and fuzzy constraints. Therefore, a satisfying solution of the following stochastic optimization problem:

$$(P_1)' \begin{cases} \max \mu_D(x,\omega) \\ x \in X \cap \text{Supp}\, \mu_D \end{cases}$$

where

$$\mu_D(x,\omega) = \min_{i=0,1\ldots,m} \mu_i(x,\omega)$$

(P_1)' can now be solved using techniques of Stochastic Optimization (A. Charnes and W.W. Cooper, 1963, S. Vryasev and P.M. Pardalos, 2010).

For instance, if one considers the expectation value approach, one has to solve the following optimization problem.

$$(P_1)'' \begin{cases} \max E(\mu_D(x,\omega)) \\ x \in X \cap \text{Supp}\, \mu_D \end{cases}$$

To handle this problem, we need an analytical expression of the distribution of $\mu_D(x,\omega)$. An interested reader is referred to E. Czogala, 1988, for details on these matters.

A part from the above symmetrical approach for dealing (P_1), there exists symmetrical approaches (M.K. Lundjula and M.M. Gupta, 1996, and F.Aiche, 1994) where the constraints serve to limit the feasible set and where the objective function is used to rank feasible alternatives.

To round out this section, let's mention the fact that the above mentioned developments where followed by systematic comparison between stochastic programming and Fuzzy Optimization (J.J. Buckley, 1990, M. Inuiguchi and M. Sakawa, 1995).

These studies displayed many similarities and differences that have been put in good use to deal with hard stochastic programs through simple and relevant fuzzy optimization techniques (S. Hursulka, M.P. Biwal and S.B. Sinha, 1997, C. Mohan and H.J. Nguyen, 1997, S.B. Sinha, S. Hursulka and M.P. Biwal, 2000) and vice versa (J.R. Rodrigues, 2005).

By the same token, approaches for considering simultaneously fuzzy and stochastic constraints in a same mathematical program were described in C. Mohan and H.J. Nguyen, 2001. Flexible programming with random data is used in several applications (S.Wang and G.H. Huang, 2011, H. Rommelfanger, 1996, T.F. Liang 2012 and John Munro, 1984).

Mathematical Program with random variables having fuzzy parameters

Mathematical Programs with random variables having fuzzy parameters are in common occurrence in many applications (S. Nanda, G. Panda and J. Dash, 2006, F. ben Abdelaziz, L.Enneifar and J.Martel, 2004).

As a matter of fact, experts who provide data for a problem that may be cast into a mathematical programming setting may feel more comfortable in coupling their vague perceptions with hard statistical data.

By the way of example, consider a portfolio selection problem where, due to stock experts' judgments and investors' different options, the security returns are modelized as random variables with imprecise parameters.

Readers interested in mathematical formulation and treatment of random variables with fuzzy parameters might refer to J.J. Buckley and E. Eslami, 2003, and J.J. Buckley and E. Eslami, 2004.

Consider the mathematical program

$$(P_2)\ \begin{cases} \min \bar{c}x \\ A_i^* \leq b_i^* \quad i = 1, \dots, m \\ x \geq 0 \end{cases}$$

where * means that the datum is a normal random variable with some fuzzy parameters.

To convert (P_2) in deterministic terms, a fuzzified version of the well-known chance-constrained programming approach (J.R. Birge and F.Louveaux, 1997) is used in the literature, (see e.g. M.K. Luhandjula, 2004 and M.K. Luhandjula, 2010).

A deterministic counterpart of (P_2) is then obtained through the following optimization problem:

$$(P_2)'\ \begin{cases} \min cx \\ \tilde{P}\ \left(\sum_{j=1}^{n} a_{ij}^* x_j \leq b_i^*\right) \geq \tilde{\delta}_i; \ \ i = 1, \dots, m \\ x \in X = \{x \in \mathbb{R}^n / x \geq 0\} \end{cases}$$

Where $\tilde{P}$ stands, for uncertain probability (J.J. Buckley and E. Eslami, 2003, J.J. Buckley and E. Eslami, 2004), $\tilde{\delta}_i$ $(i = 1, \dots, m)$ are fuzzy thresholds fixed by Decision Maker and c = $E(\bar{c})$.

Three cases may be considered.

Case 1: b_i^* $(i = 1, \dots, m)$ are real numbers denoted merely byb_i $(i = 1, \dots, m)$.

This means that for all I, b_i^* is regarded as a random variable having as set of parameters the singleton $\{b_i\}$.

It is further assumed that for all (i, j), a_{ij}^* is a normally distributed random variable with fuzzy number $\tilde{m}_{ij}$ and fuzzy number variance $\tilde{\sigma}_{ij}^2$.

In this case, $\bar{\mu}_i = \sum_{i=1}^{n} a_{ij}^* x_j$ is also a random variable whose mean and variance are fuzzy numbers denoted by $\tilde{m}_{\bar{\mu}_i}$ and $\tilde{\sigma}_{\bar{\mu}_i}^2$ respectively (H. Kwakernaak, 1979).

As $\tilde{\delta}_i$, $\tilde{m}_{\bar{\mu}_i}$ and $\tilde{\sigma}_{\bar{\mu}_i}^2$ are fuzzy numbers, their α-levels are real intervals denoted as follows:

$$\bar{\delta}_i^{\alpha} = [\delta_i^{\alpha L}, \quad \delta_i^{\alpha U}]$$

$$\tilde{m}_{\bar{\mu}_i}^{\alpha} = [m_{\bar{\mu}_i}^{\alpha L}, \quad m_{\bar{\mu}_i}^{\alpha U}]$$

$$\tilde{\sigma}_{\bar{\mu}_i}^{2\alpha} = [\, \sigma_{\bar{\mu}_i}^{2\alpha L}, \ \sigma_{\bar{\mu}_i}^{2\alpha U}].$$

The following result, the proof of which may be found in M.K. Luhandjula (2010), provides a deterministic counterpart of (P_2) through $(P_2)'$.

Theorem 3.1

If in addition to the above mentioned assumptions a_{ij}^* $(j = 1, \dots, n)$ are independent, then $(P_2)'$ is equivalent to the following optimization problem:

$$(P_2)'' \begin{cases} \min cx \\ \Phi\left(\dfrac{b_i - m_{\mu_i}^{\alpha U}}{\sigma_{\mu_i}^{2\alpha U}}\right) \geq \delta_i^{\alpha U} \quad \forall \alpha \in (0, 1]; \ \ i = 1, \dots, m \\ x \in X = \{x \in \mathbb{R}^n / \, x \geq 0\} \end{cases}$$

where Φ is the cumulative distribution of normal 0-1.

Case 2: a_{ij}^* $(i = 1, \dots, m\,;\ j = 1, \dots, n)$ are real numbers.

It is also assumed that, for all i, b_i^* is normally distributed random variable whose mean and variance are $\tilde{m}_{b_i}$ and $\tilde{\sigma}_{b_i}^2$ respectively.

In this case, the α-cuts of $\tilde{m}_{b_i}$ and $\tilde{\sigma}_{b_i}^2$ are respectively

$$\tilde{m}_{b_i}^{\alpha} = [m_{b_i}^{\alpha L} \quad , \quad m_{b_i}^{\alpha U}]$$

$$\tilde{\sigma}_{b_i}^{2\alpha} = [\sigma_{b_i}^{2\alpha L} \quad , \ \sigma_{b_i}^{2\alpha U}].$$

The following result, the proof of which may be found in M.K. Luhandjula, 2010, gives a crisp counterpart of (P_2) through $(P_2)'$.

Theorem 3.2

If, in addition to the above assumptions b_i^* $(i = 1, \dots, m)$ are independent, then (P_2)' is equivalent to the following mathematical program:

$$(P_2)''' \begin{cases} \min cx \\ \Phi\left(\frac{\sum_{j=1}^{n} a_{ij} x_i - m_{b_i}^{\alpha U}}{\sigma_{b_i}^{2\alpha U}}\right) \le 1 - \delta_i^{\alpha U} \quad \forall \alpha \in (0,1];\ i = 1, \dots, m \\ x \in X = \{x \in \mathbb{R}^n / x \ge 0\} \end{cases}$$

Case 3: General case

Here we assume that both a_{ij}^* and b_i^* are random variables with fuzzy parameters.

Let $\xi_i^* = \sum_{j=1}^{n} a_{ij}^* x_j - b_i^*$

Then according to M.K. Luhandjula, 2010, ξ_i^* is also normally distributed with fuzzy means $\tilde{m}_{\xi_i(x)}$ and fuzzy variance $\tilde{\sigma}_{\xi_i(x)}^2$.

The following result (M.K. Luhandjula, 2010) provides a crisp counterpart of (P_2) through $(P_2)'$ for the general case.

Theorem 3.3

Under the above mentioned assumptions and if a_{ij}^* and b_i^* are independent, then (P_2)' is equivalent to the following optimization problem:

$$\begin{cases} \min cx \\ \Phi\left(\frac{-m_{\xi_i(x)}^{\alpha U}}{\sigma_{\xi_i(x)}^{2\alpha U}}\right) \ge \delta_i^{\alpha U} \quad \forall \alpha \in (0,1];\ i = 1, \dots, m \\ x \in X = \{x \in \mathbb{R}^n / x \ge 0\} \end{cases}$$

From the above three results, algorithms have been described for solving (P_2). An interested reader is invited to consult M.K. Luhandjula, 2010, for details on these matters.

Mathematical programming with fuzzy random coefficients

Without a shadow of doubt, fuzzy random variable development (H.Kwakernaak, 1979, A. Colbi, J.S.Dominguez Menchero, M.Lopez-Diaz and D.A. Ralescu, 2001), has been catalyst that h

As matter of fact, fuzzy random variables provided a gold mine of opportunities for dealing with several aspects where fuzziness and randomness are combined in a mathematical setting.

One of the first optimization model involving fuzzy random coefficients is given below.

$$(P_3) \begin{cases} \min \sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} a_{ij} x_j \subseteq b_i\ ;\ i = 1, \dots, m \\ x_j \ge 0\ ;\ j = 1, \dots, n \end{cases}$$

where c_j , a_{ij} and b_i are fuzzy random variables on $(\Omega, \mathcal{F}, P)$.

A first step toward solving this problem is to put it in the following equivalent form.

$$(P_3)' \begin{cases} \min t \\ \sum_{j=1}^{n} c_j x_j \leq t \\ \sum_{j=1}^{n} a_{ij} x_j \subseteq b_i \ ; \ i = 1, \dots, m \\ x_j \geq 0 \ ; j = 1, \dots, n \end{cases}$$

where t is a maximal tolerance interval for the objective function.

It is shown in M.K. Luhandjula, (2004) that $(P_3)'$ can be put in the form of a semi-infinite stochastic program. An approach combining Monte-Carlo simulation and cutting-plane technique for semi-infinite stochastic optimization problems may be found elsewhere (M.K. Luhandjula, 2007).

For the inequality constrained case, that is for the optimization problem:

$$(P_4) \begin{cases} \min \sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \ ; \ i = 1, \dots, m \\ x_j \geq 0 \ ; j = 1, \dots, n \end{cases}$$

where c_j , a_{ij} and b_i are fuzzy random variables, the commonly used approach is to craft a deterministic surrogate of the fuzzy stochastic optimization problem at hand, by exploiting structure available while sticking, as well as possible, to uncertainty principles.

Two paradigms are used to this end: the approximation paradigm (N.Van Hop, 2007 and E.E. Ammar, 2009) and the equivalence one (M.K. Luhandjula, 2011).

In the approximation paradigm the original problem is approximated (in some sense) by another one. The later being solved by existing techniques. For instance, replace involved fuzzy random variables by their expectation and solve the resulting fuzzy program by existing techniques (Y.K.Liu and B. Liu, 2005). Approaches along this approximation paradigm have often been questioned in terms of robustness and general validity. As a matter of fact, without a serious output analysis, ascertain both the quality of the approximation and the validity of obtained solutions.

Regarding the equivalence paradigm, the problem at hand is replaced by an equivalent one. The equivalent problem is obtained by making use of an Embedding Theorem for fuzzy random variables.

To put this in perspective, consider the following optimization problem:

$$(P_5)\ \begin{cases}\min \tilde{f}(x)\\ x \in X\end{cases}$$

where $\tilde{f}: \mathbb{R}^n \longrightarrow F(\Omega)$, $F(\Omega)$ denotes the space of fuzzy random variables on $(\Omega, \mathcal{F}, P)$ and X is a convex and bounded subset of $\mathbb{R}^n$.

(P_5) is equivalent to

$$(P_5)'\ \begin{cases}\min \sigma(\tilde{f}(x))\\ x \in X\end{cases}$$

where σ is the isomorphism obtained from the Embedding Theorem for fuzzy random variables (M.K. Luhandjula, 2011).

Making use of the definition of σ, $(P_5)'$ can be written as follows.

$$(P_5)''\begin{cases}\ \min\left\{\left[\tilde{f}^L_\omega(x)(\alpha),\ \tilde{f}^U_\omega(x)(\alpha)\right]\right\}\\ x \in X\\ \alpha \in (0,1];\ \omega \in \Omega\end{cases}$$

Worthy to denote here is the fact that $(P_5)''$ is a stochastic multiobjective program with infinitely many objective interval functions. Some ways to deal with this optimization problem are described in M.K. Luhandjula and A.S. Adeyafa (2010).

Mathematical programming problem with random data and fuzzy numbers may be solved using approaches discussed in this section.

As a matter of fact, random data and fuzzy numbers may be regarded as degenerate fuzzy random variables.

An interested reader is referred to M.K. Luhandjula, 2004, where an approach for solving a linear program having fuzzy numbers as coefficients of technological matrix and random variables as components of the vector of the second member is described.

Extensions and Applications

Ideas discussed in previous sections have been extended to nonlinear programming problems in the presence of fuzzy and random data (E.E. Ammar, 2008, Y.K.Liu and B. Liu, 1992, and B.Liu, 2001). Extensions of Fuzzy Stochastic Optimization have also been carried out towards multiobjective Programming Problems (Jun Li and Jiuping Xu, 2008, M. Sakawa, I.Nishizaki and H. Katagiri, 2011, H. Katagiri, M. Sakawa and H.Ishii, 2005), multilevel optimization (R.Liang, J.Gao and K.Iwamura, 2007, and J.Gao and B. Liu, 2005) and multistage mathematical programs.

The field of Fuzzy Stochastic optimization is rich of potential applications, as a matter of fact, uncertainty and ubiquitous in real life problems. Zadeh's incompatibility principle stipulating that, when the complexity of system increases, our aptitude to make precise statements about it decreases up to threshold where precision and significance become mutually exclusive characteristics, is telling in this regard.

The simplistic way consisting of replacing arbitrarily imprecise data by precise ones, caricature badly the reality.

Many applications of Fuzzy Stochastic optimization are reported in the literature. Here are, without any claim for exhaustive study, some of them: Financial applications (Z. Zimeskal, 2001), industrial applications (H.T. Nguyen, 2005), marketing applications (K.Weber and L. Gromme, 2004), water resources applications (I. Maqsood, G.H. Huang and J.S. Yeomans, 2005) and portfolio applications (X. Huang, 2007).

Concluding remarks and perspectives for future research

In many concrete situations, one may have to combine evidence from different sources and as a result to grapple with both probabilistic and probabilistic uncertainty. The proved irreducible differences between the two kinds of uncertainty call for ways of integrating them simultaneously into mathematical models.

To assert that it is more useful to conceive imprecision as a variegated whole is not to minimize important research works that have been done in specific aspects (Fuzzy Programming, Stochastic Programming).

It is instead to assert that new perspectives for coping with complex real life problems may be gained by integration of both approaches than exclusion.

We have surveyed the terrain covered by Fuzzy Stochastic Optimization with an eye to some important themes and questions, with propensity for ideas rather than technical considerations.

The main lessons that can be drawn from this overview are as follows.

- The area of competence of Fuzzy Stochastic Optimization is known along with some matrix of values that make it distinctive from other fields of mathematical programming under uncertainty.
- Fuzzy Stochastic Optimization deserves attention of researchers.

As a matter of fact, it is of great help in pulling users of mathematical programming models out of abyss of resorting to the hammer principal (when you have only a hammer, anything at your hand is considered as a nail), while making decision in a complex environment involving both randomness and fuzziness.

A blind suppression of inherent randomness and fuzziness for the sake of data uniformization, leads generally to a caricatured picture of reality.

- The field is still in stammering stage and a lot of work remains to be done.

Among lines for further developments in this field, we may mention:

- Theoretical contributions on the characterization of solutions of Fuzzy Stochastic programs.
- Production of user-friendly software for Fuzzy Stochastic Optimization Problems.
- Design of epistemological choices for defuzzification and derandomization that leads to effective and efficient approaches for solving fuzzy stochastic optimization problems.
- Description of high quality case studies in order to demonstrate usefulness of Fuzzy Stochastic Optimization.

References

Aiche F., (1994). Sur l'Optimisation floue stochastique. Dissertation. University of Tizi-Ouzou,

Ammar E.E., (2008). On solutions of fuzzy random multiobjective quadratic programming with applications in portfolio problem. Information Sciences 178, 468-484.

Ammar E.E., (2009). On fuzzy random multiobjective quadratic programming. European Journal of Operational Research 193, 329-341.

Avriel M., (1976). Nonlinear programming . Prentice- Hall.

Ben Abdelaziz F., Enneifar L, Martel J., (2004). A multiobjective fuzzy stochastic program for water resources optimization: The case of lake management. International System and Operational Research Journal 42, 201-215.

Birge J.R, Louveaux F., (1997) Introduction to Stochastic Programming. Springer, New York.

Buckley J.J., (1990). Stochastic versus possibilistic programming. Fuzzy Sets and Systems 34, 173-177.

Buckley J.J., (1990). Stochastic versus possibilistic multiobjective programming. In Slowinski and Teghem (Eds), Stochastic versus fuzzy approaches to Multiobjective Mathematical Programming under Uncertainty, 353-364.

Buckley J.J., Eslami E., (2003). Uncertain probabilities I: The discrete case. Soft Computing 7, 500-505.

Buckley J.J., Eslami E., (2004). Uncertain probabilities II: The continuous case. Soft Computing 8, 193-199.

Buckley J.J., Eslami E.,(2004). Uncertain probabilities III: The continuous case. Soft Computing 8, 200-206.

Charnes A., Cooper W.W., (1963) Deterministic equivalents for optimizing and satisfying under uncertainty. Operational Research 11, 18-39.

Colubi A., Dominguez-Menchero J.S., Lopez-Diaz M., Ralescu D.A., (2001). On the formalization of fuzzy random variables, Information Sciences 133, 3-6, 2001.

Czogala E., (1988). On the choice of optimal alternatives for decision making in probability fuzzy environment. Fuzzy Sets and Systems 28, 131-141.

Dubey D., Chandra S., (2012). Fuzzy linear programming under interval uncertainty based on IFS representation. Fuzzy Sets and Systems 188, 68-87.

Dubois D., (2011).The role of fuzzy sets in decision sciences: Old techniques and new directions. Fuzzy Sets and Systems 184, 3-28.

Dubois D., Prade H., (1980). Fuzzy Sets and Systems. Prentice Hall.

Eaves B.C., Zangwill W.I., (1971). Generalized Cutting Plane Algorithms. SIAM Journal of Control 9, 529-542

Gao J. and Liu B., (2005). Fuzzy multilevel programming with a hybrid intelligent algorithm. Computers and Mathematics with Applications 49, 1539-1548.

Gass S.I., (1985). Linear Progrmming methods and applications. Dover Publications.

Gentle J., Härdle W., Mari Y. (Eds), (2004). Handbook of Computational Statistics. Springer.

Hasuike T., Ishii H., (2004). Probability maximization models for portfolio selection under ambiguity. Central European Journal of Operations Research 17, 159-180.

Hirota K.,(1981). Concepts of probabilisticsets. Fuzzy Sets and Systems 5, 31-46.

Huang X., (2007). Two new models for portfolio selection with stochastic returns. European Journal of Operational research 180, 396-405, 2007.

Hulsurka S., Biwal M.P., Sinha S.B., (1997). Fuzzy programming approach to multiobjective stochastic linear programming problems. Fuzzy Sets and Systems 88, 173-181

Inuiguchi M., Sakawa M., (1995). A possibilistic linear program is equivalent to a stochastic linear program in a special case. Fuzzy Sets and Systems 76, 309-317.

Kall P., Wallace S.W.,(1994). Stochastic Programming. John Wiley.

Katagiri H., Sakawa M., Ishii H., (2005). A study on fuzzy random portfolio selection problems using possibility and necsssity measures. Scientiae Mathematicae, Japonocae 61, 361-369.

Khachiyan L.G., (1979). A polynomial algorithm for linear programming. Soviet Mathematics Doklady, vol. 20, 191-194.

Kwakernaak H., (1979). Fuzzy random variables I: Definitions and Theorems. Information Sciences 15, 1-29.

Li Jun, Jiuping Xu, (2008). Compromise solution for constrained multiobjective portfolio selection in uncertain environment. Mathematical Modeling and Computing.

Liang T.F., (2012). Integrated manufacturing / distribution planning decisions with multiple imprecise goals in an uncertain environment. International Journal of Methodology 46, 137-153.

Liang,R., Gao J. and Iwamura K., (2007). Fuzzy random dependent-chance bilevel programming with applications. In Advances in Neural Networks, Liu, Fei, Hou, Zhang and Sun (Eds). Springer 257-266.

Liu B., (2001). Fuzzy random chance constrained programming. IEEE Transactions on Fuzzy Systems 9, 713-720.

Liu Y.K., and Liu B., (1992). A class of fuzzy random optimization: Expected value models. Information Sciences 155, 89-102.

Liu Y.K., and Liu B., (2005). Fuzzy random programming with equilibrium chance constraints. Information Sciences 170, 363-395.

Luhandjula M.K. and Gupta M.M., (1996). On Fuzzy Stochastic Optimization. Fuzzy Sets and Systems 81, 47-55.

Luhandjula M.K., (1983). Linear programming under randomness and fuzziness. Fuzzy Sets and Systems 10, 45-55.

Luhandjula M.K., (1989). Fuzzy Optimization: An appraisal, Fuzzy Sets and Systems 30, 257-282.

Luhandjula M.K., (2004). Optimization under hybrid uncertainty. Fuzzy Sets and Systems 146, 187-203.

Luhandjula M.K., (2007). A monte Carlo simulation based approach for stochastic semi-infinite mathematical programming problems. International Journal of uncertainty, Fuzziness and Knoweledge-based Systems 15, 139-157.

Luhandjula M.K., (2011). On Fuzzy Random-Valued Optimization. American Journal of Operations Research 1, 259-267.

Luhandjula,M.K and Joubert J.W., (2010). On some optimization models in a fuzzy-stochastic environment. European Journal of Operational Research 207, 1433-1441.

Luhandula M.K. , Adeyefa A.S., (2010). Multiobjective programming problems with fuzzy random coefficients. Advances in Fuzzy Sets and Systems 7, 1-16.

Maqsood I., Huang G.H., Yeomans J.S., (2005). An interval-parameter fuzzy two-stage stochastic program for water resource management under uncertainty. European Journal of Operational Research 167, 208-225.

Mohan C., Nguyen H.J., (1997) A fuzzifying approach to Stochastic programming. Operational Research 34, 73-96.

Mohan C., Nguyen H.J., (2001). An interactive satisfying method for solving mixed fuzzy stochastic programming problems. Fuzzy Sets and Systems 117, 67-79.

Munro J., (1984). Fuzzy programming and imprecise data. Civil Engineering Systems 1, 255-260.

Nanda S., Panda G., Dash J., (2006) .A new solution method for fuzzy chance constrained programming problem. Fuzzy Optimization and Decision Making 5, 355-370.

Nguyen H.T., (2005). Optimization in fuzzy stochastic environment and its applications in industry and economics. Available from: www.haul.edu.vn.

Rodrigues J.R., (2005) Optimization under fuzzy if-then rules using stochastic algorithms. In: Puigjaner and Espuña (Eds). European Symposium on Computer Aided Process Engineering.

Rommelfanger H., (1996).Fuzzy linear programming and its applications. European Journal of Operational Research 92, 512-527.

S.B. Sinha, S. Hulsurka, M.P. Biwal,(2000). Fuzzy programming approach to multiobjective stochastic programming problems. Fuzzy Sets and Systems 109, 91-96.

Sakawa M., Nishizaki I., Katagiri H., (2011). Fuzzy Stochastic Multiobjective Programming. Springer.

Sengupta J.K, (1972). Stochastic Programming: Methods and Applications. North Holland.

Vajda S., (1972). Probabilistic Programming, Academic Press, New-York.

Van Hop N., (2007). Solving fuzzy stochastic linear programming problems using superiority and inferiority measures. Information Sciences 177, 1977-1991.

Van Hop N., (2007). Solving linear programming problems under fuzziness and randomness environment using attainment values. Information Sciences 177, 2971-2984.

Vasant P., Nagarajan R., Yaacob S., (2005). Fuzzy linear programming: A modern tool for Decision making. Studies in Computational Intelligence 2, 383-401.

Vryasev S. and Pardalos P.M., (2010). Stochastic Optimization: Algorithms and Applications. Kluwer.

Wang S. and Huang G.H., (2011). Interactive two-stage stochastic programming for water resources management. Journal of Environmental Management 92, 1986-1995.

Wang S., Huang G.H., Yang B.T., (2012). An interval-valued fuzzy-stochastic programming approach and its applications to municipal solid waste management. Environmental Modelling § Software 29.

Wang S., Watada J., (2012). Fuzzy Stochastic Optimization: Theory, Models and Applications. Springer.

Weber K., Gromme L., (2004). Decision Support System in marketing by means of fuzzy stochastic optimization. GAMM.

Zmeskal Z., (2001). Application of fuzzy-stochastic methodology to appraising the firm value as European call option. European Journal of Operational Research, 135, 303-310.